TRIADS OF TRANSFORMATIONS OF CONJUGATE SYSTEMS OF CURVES

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When the rectangular point coordinates x,y,z, of a surface satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial u\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{1}$$

the curves $u = \text{const.}$ $v = \text{const.}$ form a conjugate system. We assume that the parametric system is of this sort throughout this note, and we shall speak of the *net* of parametric curves. Equation (1) is the *point equation* of the net.

If N is such a net, a second net N' of coordinates x', y', z', is given by the quadratures

$$\frac{\partial x'}{\partial u} = h'\frac{\partial x}{\partial u}, \qquad \frac{\partial x'}{\partial v} = l'\frac{\partial x}{\partial v}, \tag{2}$$

provided that h' and l' are functions of u and v subject to the conditions

$$\frac{\partial h}{\partial v} = a\,(l' - h'), \qquad \frac{\partial l'}{\partial u} = b\,(h' - l'). \tag{3}$$

Moreover, each pair of solutions of these equations leads by (2) to a net N', which is such that the tangents at corresponding points M and M' to the curves of the nets are parallel. All nets *parallel* to N are obtained in this way.

If θ_1 is any solution of (1), and θ_1' is the function given by

$$\frac{\partial\theta_1'}{\partial u} = h'\frac{\partial\theta_1}{\partial u}, \qquad \frac{\partial\theta_1'}{\partial v} = l'\frac{\partial\theta_1}{\partial v}, \tag{4}$$

then the functions $x_1^{(1)}$, $y_1^{(1)}$, $z_1^{(1)}$, defined by equations of the form

$$x_1^{(1)} = x - \frac{\theta_1}{\theta_1'}x' \tag{5}$$

are the coordinates of a net $N_1^{(1)}$, so related to N that the lines joining corresponding points M and $M_1^{(1)}$ of these nets form a congruence whose developables meet the surface on which these nets lie in the curves of the nets. We say that two nets so related geometrically are in the relation of a *transformation* T. Parallel nets are in such relation. We have shown[1] that any transformation T of N into a non-parallel net is given by equations of the form (5).

Let N'' be a second net parallel to N, its coordinates being given by

$$\frac{\partial x''}{\partial u} = h'' \frac{\partial x}{\partial u}, \qquad \frac{\partial x''}{\partial v} = l'' \frac{\partial x}{\partial v}, \tag{6}$$

and let θ_1'' be defined by

$$\frac{\partial \theta_1''}{\partial u} = h'' \frac{\partial \theta_1}{\partial u}, \qquad \frac{\partial \theta_1''}{\partial v} = l'' \frac{\partial \theta_1}{\partial v}, \tag{7}$$

h'' and l'' being a pair of solutions of (3).

Then a second transform $N_1^{(2)}$ has coordinates of the form

$$x_1^{(2)} = x - \frac{\theta_1}{\theta_1''} x''. \tag{8}$$

Since the nets N' and N'' are parallel to one another, and the functions θ_1' and θ_1'' are solutions of the respective point equations for N' and N'' in a relation analogous to (4), a transformation T of N'' is given by

$$\overset{\prime\prime\prime}{x}{}_1^{(1)} = x'' - \frac{\theta_1''}{\theta_1'} x'. \tag{9}$$

We denote by $\overset{\prime\prime\prime}{N}{}_1^{(1)}$ the net with these coordinates. By differentiating the expressions (9), we show that the nets $N_1^{(1)}$ and $\overset{\prime\prime\prime}{N}{}_1^{(1)}$ are parallel. Moreover, it can be shown that the equations

$$x_1^{(2)} = x_1^{(1)} - \frac{\theta_1}{\theta_1''} \overset{\prime\prime\prime}{x}{}_1^{(1)} \tag{10}$$

are consistent with the above equations, and consequently $N_1^{(2)}$ is a T transform of $N_1^{(1)}$. Hence *if a net is transformed into two nets by means of the same function θ, the new nets are in the relation of a transformation T.* We say that three such nets form a *triad* under transformations T. It can be shown that the relation is entirely reciprocal in the sense that any two are obtainable from the third by transformations involving the same solution of the point equation of the third net. If in particular we take for θ_1 any of the coordinates of N, say z, the nets $N_1^{(1)}$ and $N_1^{(2)}$ lie in the plane $z = 0$. In other words, the developables of the two congruences, obtained by drawing through points of a net N lines parallel to the corresponding radii vectores of two nets parallel to N meet any plane in two nets in the relation of a transformation T. (We postpone to a later time a discussion of transformations of planar nets.)

If θ_i is any solution of (1), a solution of the point equation of $N_1^{(1)}$ is given by

$$\theta_{12} = \theta_2 - \frac{\theta_1}{\theta_1'}\,\theta_2'. \tag{11}$$

This function and the net $\overset{'''}{N}{}_1^{(1)}$ parallel to $N_1^{(1)}$ determine a transformation of the latter; moreover, the congruence of the transformation consists of the joins of corresponding points on $N_1^{(1)}$ and $N_1^{(2)}$. We call the transform N_{12} and its point coordinates x_{12}, y_{12}, z_{12}. The solution of the point equation of $\overset{'''}{N}{}_1^{(1)}$ corresponding to θ_{12} is of the form

$$\theta_{12}''' = \theta_2'' - \frac{\theta_1''}{\theta_1'}\,\theta_2', \tag{12}$$

and consequently we have

$$x_{12} = x_1^{(1)} - \frac{\theta_2\theta_1' - \theta_1\theta_2'}{\theta_2''\theta_1' - \theta_1''\theta_2'}\,\overset{'''}{x}{}_1^{(1)}. \tag{13}$$

The function θ_2 can be used to determine with the nets N' and N'' two transforms of N, namely $N_2^{(1)}$ and $N_2^{(2)}$, whose coordinates are of the respective forms

$$x_2^{(1)} = x - \frac{\theta_2}{\theta_2'}\,x', \qquad x_2^{(2)} = x - \frac{\theta_2}{\theta_2''}\,x''. \tag{14}$$

Corresponding points of the nets N, $N_1^{(2)}$, and $N_2^{(2)}$ lie on a line, and $N_2^{(2)}$ is a transform of $N_1^{(2)}$ by means of the function

$$\theta_2 - \frac{\theta_1}{\theta_1''}\,\theta_2''. \tag{15}$$

Likewise, corresponding points of the nets $N_1^{(1)}$, $N_1^{(2)}$, N_{12} lie on a line, and N_{12} is a transform of $N_1^{(2)}$ by means of the function

$$\theta_{12} - \frac{\theta_1\theta_{12}''}{\theta_1''}. \tag{16}$$

By means of (11) and (12) we show that the expressions (15) and (16) are equal, and consequently the nets $N_1^{(2)}$, $N_2^{(2)}$, N_{12} form a triad.

Equation (13) is reducible to

$$x_{12} = x + [(\theta_1''\,\theta_2 - \theta_2''\theta_1)x' + (\theta_2^{1}\theta_1 - \theta_1'\theta_2)x'']/(\theta_1'\theta_2'' - \theta_1''\theta_2') \tag{17}$$

From the symmetry of this expression we see that N and N_{32} are transforms of $N_1^{(1)}$ and $N_2^{(2)}$ in an analogous manner. We say that the four nets form a *quatern* $(N, N_1^{(1)}, N_2^{(2)}, N_{12})$. This result constitutes a generalization of the theorem of permutability of transformations D_m of isothermic surfaces as established by Bianchi[2]. In like manner we have the quaterns $(N, N_1^{(2)}, N_2^{(1)}, N_{12})$ and $(N_2^{(1)}, N_1^{(1)}, N_2^{(2)}, N_1^{(2)})$. Moreover the six nets can be associated into the four triads $N, N_1^{(1)}, N_1^{(2)}$; $N, N_2^{(1)}, N_2^{(2)}$; $N_1^{(2)}, N_2^{(2)}, N_{12}$; $N_2^{(1)}, N_1^{(2)}, N_{12}$.

When the nets $N_1^{(1)}$ and $N_2^{(2)}$ have been found, the functions θ_1'' and θ' are determined to within additive constants. Hence, *if $N_1^{(1)}$ and $N_2^{(2)}$ are two transforms of N, there exist ∞^2 nets N_{12}, each of which forms a quatern with N, $N_1^{(1)}$ and $N_2^{(2)}$; and their determination requires two quadratures.*

The six corresponding points of the nets are the vertices of a complete quadrilateral whose four sides are generic lines of the four congruences which figure in the transformations. On each of these lines there are two focal points, each being the point of contact of the line with the edge of regression of one or other of the two developables as u or v varies. The four points corresponding to the variation of either variable lie on a line, and these two lines are the tangents to the parametric curves on the envelope of the plane of the quadrilateral; moreover, these curves form a net.

Thus far we have used rectangular non-homogenous point coordinates, but in some cases it is advisable to make use of general homogenous coordinates. The four homogenous coordinates x, y, z, w, of a net satisfy an equation of the form.

$$\frac{\partial^2\theta}{\partial u\partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v} + c\theta. \tag{18}$$

When two nets N and N_1 are in the relation of a transformation T, the tangents to the curves v = const. at corresponding points M and M_1 of the net meet in a point F_1. Likewise the tangents at M and M_1 to the curves u = const. meet in a point F_2. It is readily seen that as v varies, any point on the tangent to a curve v = const. of a net moves in such a way that the tangent to its path lies in the tangent plane of the net. Similarly for a point on the tangent to u = const., as u varies. Since the line F_1F_2 lies in the tangent planes to both N and N_1, it is tangent to the motion of F_1 as v varies, and to the motion of F_2 as u varies. Hence F_1 and F_2 are the focal points of the congruence of lines of intersection of the planes of the nets. Following Guichard, we say that a congruence whose focal points lie on the tangents to the curves of a net and whose developables correspond to the curves of the net is *harmonic* to the net. It can readily be shown that the homogenous coordinates of a net can be chosen so that the coordinates of the focal points of a harmonic congruence are of the respective forms

$$\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}. \tag{19}$$

In this case equation (18) assumes the form (1), so that the choice of coordinates referred to its equivalent to finding a particular solution of (18).

Since the congruence F_1F_2 is harmonic to both N and N_1, it follows that the equations of any transformation T in homogenous coordinates is reducible to the form (2). If the coordinates of N satisfy (18), the equations are of the form

$$\frac{\partial x_1}{\partial u} = h\frac{\partial}{\partial u}\left(\frac{x}{\theta}\right), \qquad \frac{\partial x_1}{\partial v} = l\frac{\partial}{\partial v}\left(\frac{x}{\theta}\right), \tag{20}$$

where now θ is a solution of (18).

When the equations of the transformation are of the form (2), each solution of the point equation of N gives a new transform by means of (5). The equations of the preceding results continue to be true, and parallel nets are replaced by any transforms.

Although these results have been stated in terms of 3-space, they hold for two dimensional spreads in n-space, provided that a congruence is defined as a two parameter family of lines possessing two families of developables.

[1] Eisenhart, *Trans. Amer. Math. Soc., New York*, **18**, 1917, (97–124).
[2] Bianchi, *Ann. Mat., Milano*, (Ser. 3), **11**, 1905, (93–158).

THE MOLECULAR WEIGHTS OF THE TRIARYLMETHYLS

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It is now generally accepted that the free radicals of the triphenylmethane series owe their unique unsaturated character to the presence of a trivalent carbon atom in the molecule. In many cases the molecular weight has been found to be double that calculated for the free radical. Nevertheless, even in these cases the presence of a compound with a single unsaturated carbon atom is still recognized, and the assumption is made that there exists, in virtue of partial dissociation, a mobile equilibrium:

$$R_3C-CR_3 \rightleftarrows R_3C+R_3C.$$